# Sufficient conditions for superstability of many-body interactions

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**Abstract**

A detailed analysis of necessary conditions on a family of many-body potentials, which ensure stability, superstability or strong superstability of a statistical system is given in present work.There has been given also an example of superstable many-body interaction.

**Keywords :** Continuous classical system; many-body interaction; criteria of superstability.

**Mathematics Subject Classification :** 82B05; 82B21

## 1 Introduction

During last 30 years different sufficient conditions and restrictions on 2-body potential, which imply superstable or strong superstable interaction have been studied(see [8] for survey of the results). It is obvious, that the research of systems with respect to many-body interaction requires the same conditions on potential energy of interaction of any finite quantity of particles to be fulfilled. In accordance with this fact, one has a similar problem to describe the necessary conditions on a sequence of p-body ($p \geq 3$) potentials, which ensure stability, superstability or strong superstability of an infinite statistical system. We have to mention, that such conditions(which ensure an existence of corellation function in the thermodynamic limit) have been written in rather abstract form in the works [2], [3] and more implicitly in the works [4], [5], [6], [7], [10], [11]. There is another interesting work in this field(see [1] ), in which authors consider a finite sequence of finite range many-body potentials, one of which is *stabilizing*, and ensures stability of a whole system. In the present paper we consider an infinite system of finite range many-body potentials taking into account the traditional concept, i.e. in some sense p-body potential plays less important role in the total energy of interaction than $p-1$-one. Each of p-body potentials can be both positive or negative and it depends on the configuration of particles. The

conditions on a sequence of p-body ($p > 2$) potentials, which ensure stability, superstability or strong superstability of a system, if such a behavior is enabled by 2-body(pair) potential of interaction are formulated in this article. In the next section we give necessary definitions and formulate main result. In section 4 we give an example of many-body interaction, which yields above mentioned conditions

# 2 Definitions and main result

Let $\mathbb{R}^d$ be a d-dimensional Euclidean space. Following [9] for each $r \in \mathbb{Z}^d$ and $\lambda \in \mathbb{R}_+$ we define an elementary cube with a rib $\lambda$ and center $r$:

$$\Delta_\lambda(r) = \left\{ x \in \mathbb{R}^d \ \mid \ \lambda\left(r^i - 1/2\right) \leq x^i < \lambda\left(r^i + 1/2\right) \right\} \tag{2.1}$$

We will sometimes write $\Delta$ instead of $\Delta_\lambda(r)$, if a cube $\Delta$ is considered to be arbitrary and there is no reason to emphasize that it is centered in the particular point $r \in \mathbb{Z}^d$. We denote by $\overline{\Delta_\lambda}$ the corresponding partition of $\mathbb{R}^d$ into cubes $\Delta$. Let us consider a general type of many-body interaction specified by a family of p-body potentials $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$ and define also positive and negative parts of interaction potential:

$$V_p^+\left(x_1, \ldots, x_p\right) := \max\left\{0;\ V_p\left(x_1, \ldots, x_p\right)\right\},$$

$$V_p^-\left(x_1, \ldots, x_p\right) := \min\left\{0;\ V_p\left(x_1, \ldots, x_p\right)\right\}$$

We assume for the family of potentials $V := \left\{V_p\right\}_{p\geq 2}$ the following conditions:

**A1. Symmetry.** For any $p \geq 2$, any $\left(x_1, \ldots, x_p\right) \in \left(\mathbb{R}^d\right)^p$ and any permutation $\pi$ of the numbers $\{1, \ldots, p\}$:

$$V_p\left(x_1, \ldots, x_p\right) = V_p\left(x_{\pi(1)}, \ldots, x_{\pi(p)}\right).$$

**A2. Translation invariance.** For any $p \geq 2$, any $\left(x_1, \ldots, x_p\right) \in \left(\mathbb{R}^d\right)^p$ and $a \in \mathbb{R}^d$:

$$V_p\left(x_1, \ldots, x_p\right) = V_p\left(x_1 + a, \ldots, x_p + a\right).$$

**A3. Repulsion for small distances.** There exists a partition of $\mathbb{R}^d$ into cubes $\overline{\Delta_\lambda}$ (see (2.1)) such that for any $\left(x_1, \ldots, x_p\right) \subset \Delta,\ p \geq 2$: $V_p\left(x_1, \ldots, x_p\right) \geq 0$.

**A4. Integrability.**

$$\sup_{\{x_1,\ldots,x_k\}\in(\mathbb{R}^d)^k} \int\limits_{\left(\mathbb{R}^d\right)^{p-k}} \left|V_p^-\left(x_1, \ldots, x_p\right)\right| \, dx_{k+1} \cdot \ldots \cdot dx_p < +\infty\,, \quad 1 \leq k \leq p-1. \tag{2.2}$$

Under assumptions **A1-A4** we introduce the energy $U(\gamma): \quad \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$, which corresponds to the family of potentials $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$ and which is defined by:

$$U(\gamma) = \sum_{p\geq 2} \sum_{\{x_1,\ldots,x_p\}\subset\gamma} V_p(x_1, \ldots, x_p)\,, \gamma \in \Gamma_0,\ |\gamma| \geq 2, \tag{2.3}$$

where $\Gamma_0$ is the space of finite configurations

$$\Gamma_0 = \coprod_{n\in\mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \left\{\gamma \subset \mathbb{R}^d \mid |\gamma| = n\right\}, \ \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \ \Gamma^{(0)} = \emptyset. \tag{2.4}$$

Let's consider also the part of a total energy, defined only by p-body potential:

$$U^{(p)}(\gamma) = \sum_{\{x_1,\dots,x_p\}\subset\gamma} V_p(x_1,\dots,x_p), \ \gamma\in\Gamma_0, \ |\gamma|\geq 2. \tag{2.5}$$

We introduce 3 kinds of interactions, defined by the family of potentials $V := \{V_p\}_{p\geq 2}$.

**Definition 1.** *Interaction, defined by the family of potentials $V := \{V_p\}_{p\geq 2}$ is called:*
*a) stable, if there exists B>0 such that:*

$$U(\gamma) \geq -B|\gamma|, \ \textit{for any}\ \gamma\in\Gamma_0; \tag{2.6}$$

*b) superstable, if there exist $A>0$, $B\geq 0$ and partition into cubes $\overline{\Delta_\lambda}$ such that:*

$$U(\gamma) \geq A\sum_{\Delta\in\overline{\Delta_\lambda}} |\gamma_\Delta|^2 - B|\gamma|, \ \textit{for any}\ \gamma\in\Gamma_0; \tag{2.7}$$

*c) strong superstable, if there exist $A>0,\ B\geq 0,\ m>2$ and partition into cubes $\overline{\Delta_\lambda}$ such that:*

$$U(\gamma) \geq A\sum_{\Delta\in\overline{\Delta_\lambda}} |\gamma_\Delta|^m - B|\gamma|, \ \textit{for any}\ \gamma\in\Gamma_0. \tag{2.8}$$

In the above conditions constants $A, B$ can depend on $\overline{\Delta_\lambda}$ and consequently on $\lambda$. In our future estimates we will use several notations, which we introduce below.

**Definition 2.** *Let $\Delta,\ \Delta_i\in\overline{\Delta_\lambda}$; $n, m, k_i, k\in\mathbb{N}, k_1+\dots+k_n = p,\ p\geq 2$. Then:*

*a)* $$I_p^{k_1,\dots,k_n}(\Delta_1,\dots,\Delta_n) := \sup_{\left\{x_1^{(1)},\dots,x_{k_1}^{(1)}\right\}\subset\Delta_1,\dots,\left\{x_1^{(n)},\dots,x_{k_n}^{(n)}\right\}\subset\Delta_n} \left|V_p^-\left(x_1^{(1)},\dots,x_{k_n}^{(n)}\right)\right|, \tag{2.9}$$

*b)* $$I_p^{k|m}(\Delta) := \sum_{(\Delta_1,\dots,\Delta_m)\subset\overline{\Delta_\lambda}} I_p^{k,\overbrace{1,\dots,1}^{m}}(\Delta,\Delta_1,\dots,\Delta_m), \ k+m = p. \tag{2.10}$$

*The sum in (2.10) means independent sums w.r.t. every $\Delta_i$, $i = \overline{1,m}$.*

**Definition 3.** *Under the conditions of the def. 2 let $\Delta_i\neq\Delta_j$, if $i\neq j$, $\Delta_i\neq\Delta$, $1\leq i\leq m$. This means that all cubes are different. Then:*

$$I_p^{k|\{k_1,\dots,k_m\}}(\Delta) := \sum_{\{\Delta_1,\dots,\Delta_m\}\subset\overline{\Delta_\lambda}} \sideset{}{'}\sum_{\pi\in P_m} I_p^{k,k_{\pi(1)},\dots,k_{\pi(m)}}(\Delta,\Delta_1,\dots,\Delta_m), \tag{2.11}$$

*where $P_m$ is a set of all permutations of numbers {1,...,m}, but the sum $\sum'_{\pi\in P_m}$ takes into account only different permutations of numbers $\{k_1,\dots,k_m\}$ (for example if $k_i = k_j$ for some $i, j$, permutation of numbers $k_i, k_j$ is considered only once).*

There are three useful remarks and two lemmas, which are closely connected with sums (2.10), (2.11)

**Remark 1.** *From the above definitions the following inequality holds:*

$$I_p^{k|\{k_1,\dots,k_i,\dots,k_m\}}(\Delta) = I_p^{k_i|\{k_1,\dots,k,\dots,k_m\}}(\Delta).$$

**Remark 2.** *If* $\lambda \to 0$*:*

$$\lambda^{md} I_p^{k|m}(\Delta) \to \sup_{\{x_1,\cdots,x_k\}\subset\Delta} \int_{R^m} \left|V_p^-(x_1,\dots,x_k,x_{k+1}\dots,x_{k+m})\right| dx_{k+1}\cdot\ldots\cdot dx_{k+m}. \quad (2.12)$$

*It allows us to write an estimate for the value of* $I_p^{k|m}(\Delta)$*.*

**Remark 3.** *Due to the assumption* **A2** *value of* $I_p^{k|m}(\Delta)$ *does not depend on the position of cubes* $\Delta$*, so we can put*

$$I_p^{k|m}(\Delta) = I_p^{k|m}. \quad (2.13)$$

**Lemma 1.** *For any* $p \geq 2$ *the following inequality holds:*

$$\sum_{j=2}^{p} \sum_{\substack{k_l\geq 1,\ 1\leq l\leq j,\\ k_1+\dots+k_j=p,\\ k_1\leq\ldots\leq k_j}} I_p^{k_1\,|\,\{k_2,\dots,k_j\}}(\Delta) \leq I_p^{1\,|\,p-1}(\Delta). \quad (2.14)$$

*Proof.* Using the definition (2.10) we can rewrite $I_p^{1\,|\,p-1}(\Delta)$ in the following form:

$$I_p^{1\,|\,p-1}(\Delta) = \sum_{(\Delta_2,\dots,\Delta_p)\subset\overline{\Delta_\lambda}} I_p^{\overbrace{1,\dots,1}^{p}}(\Delta,\Delta_2,\dots,\Delta_p) \quad (2.15)$$

The sum in the r.h.s of (2.15) can be rewritten in the form of sums over sets of nonintersecting cubes $\{\Delta_2,\dots,\Delta_j\}$, $j=\overline{2,p}$, which belong to the area $\overline{\Delta_\lambda}\setminus\{\Delta\}$. Then, neglecting some combinatoric coefficients, which are greater then unity and as $I_p^p(\Delta)\equiv 0$ for sufficiently small $\lambda$ (see **A3** and Eq. (2.9)), equation (2.15) can be represented in the form of inequality:

$$I_p^{1\,|\,p-1}(\Delta) \geq \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1,\ 1\leq l\leq j,\\ k_1+\dots+k_j=p,\\ k_1\leq\ldots\leq k_j}} \sum_{\{\Delta_2,\dots,\Delta_j\}\subset\overline{\Delta_\lambda}\setminus\{\Delta\}} \sum_{\pi\in P_j}{}' \, I_p^{k_{\pi(1)},\dots,k_{\pi(j)}}(\Delta,\Delta_2,\dots,\Delta_j) \quad (2.16)$$

Let us take into account the following obvious estimate:

$$\sum_{\pi\in P_j}{}' \, I_p^{k_{\pi(1)},\dots,k_{\pi(j)}}(\Delta,\Delta_2,\dots,\Delta_j) \geq \sum_{\pi\in P_{j\setminus\{1\}}}{}' \, I_p^{k_1,k_{\pi(2)}\dots,k_{\pi(j)}}(\Delta,\Delta_2,\dots,\Delta_j), \quad (2.17)$$

where $P_{j\setminus\{1\}}$ is a set of all permutations of numbers $\{2,\ldots,j\}$. Using (2.11), (2.16), (2.17), we obtain finally:

$$I_p^{1\,|\,p-1}(\Delta) \geq \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1,\, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\ldots\leq k_j}} I_p^{k_1\,|\,\{k_2,\ldots,k_j\}}(\Delta).$$

■

**Lemma 2.** *For any $p\geq 2$ the following inequality holds:*

$$\sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1,\, 1\leq r\leq j}} \sideset{}{'}\sum_{\pi\in P_j} I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p \leq j \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}} |\gamma_\Delta|^p I_p^{k_1|\{k_2,\ldots,k_j\}}(\Delta). \quad (2.18)$$

*Proof.* From the def. 3 (see (2.11)) and taking into account, that if we split the first sum in (2.18) into $j$ independent sums over $\Delta_i\in\overline{\Delta_\lambda}$, $i=\overline{1,j}$; $\Delta_l\neq\Delta_k$, if $l\neq k$ the number of terms increases in $j!$ times, so

$$L := \sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1,\, 1\leq r\leq j}} \sideset{}{'}\sum_{\pi\in P_j} I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p =$$

$$= \frac{1}{j!}\sum_{\substack{\Delta_1\in\overline{\Delta_\lambda},\ldots,\Delta_j\in\overline{\Delta_\lambda}\\ \Delta_l\neq\Delta_k,\, l\neq k}} \sideset{}{'}\sum_{\pi\in P_j} I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p. \quad (2.19)$$

For any $\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda}$ the following estimate is true:

$$\sideset{}{'}\sum_{\pi\in P_j} I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j) \leq \sum_{t=1}^{j}\sideset{}{'}\sum_{\pi\in P_j\setminus\{t\}} I_p^{k_t,k_{\pi(2)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j). \quad (2.20)$$

We obtain from (2.19), (2.20):

$$L \leq \frac{1}{j!}\sum_{r=1}^{j}\sum_{\Delta_r\in\overline{\Delta_\lambda}}\sum_{\substack{\Delta_1,\ldots,\Delta_{r-1}\in\overline{\Delta_\lambda},\\ \Delta_{r+1},\ldots,\Delta_j\in\overline{\Delta_\lambda},\\ \Delta_l\neq\Delta_k,\, l\neq k}}\sum_{t=1}^{j}\sideset{}{'}\sum_{\pi\in P_j\setminus\{t\}} I_p^{k_t,k_{\pi(2)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)|\gamma_{\Delta_r}|^p. \quad (2.21)$$

As the number of sets $\{\Delta_1,\ldots,\Delta_{r-1},\Delta_{r+1},\ldots,\Delta_j\}\subset\overline{\Delta_\lambda}$ in the third group of sums in (2.21) is $(j-1)!$ and taking into account the def. 3 (see(2.11)) one can rewrite (2.21) in the following way:

$$L \leq \frac{1}{j}\sum_{r=1}^{j}\sum_{\Delta_r\in\overline{\Delta_\lambda}}\sum_{t=1}^{j} I_p^{k_t|\{k_1,\ldots,k_{t-1},k_{t+1},\ldots,k_j\}}(\Delta_r)\,|\gamma_{\Delta_r}|^p. \quad (2.22)$$

We deduce finally from the remarks 1,3 and (2.22):

$$L \leq j \sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq 1}} |\gamma_\Delta|^p I_p^{k_1|\{k_2,\ldots,k_j\}}(\Delta).$$

■

We give the following definition for the positive part of interaction potential:

$$V_p^p(\Delta) := \inf_{\{x_1,\ldots,x_p\} \subset \Delta} V_p^+(x_1,\ldots,x_p) \tag{2.23}$$

The main result of the article is in the following theorem:

**Theorem 2.1.** *Let the family of* p*-body potentials* $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$ *satisfy assumptions* **A1-A4**. *Let also the part of interaction* $U^{(2)}(\gamma)$ *be stable (superstable, strong superstable). If there exists such partition of* $\mathbb{R}^d$ *into cubes* $\overline{\Delta_\lambda}$, *that for each* $p > 2$ *the following holds:*

$$1) \frac{V_p^p(\Delta)}{p^p} - p\, I_p^{1|p-1}(\Delta) \geq 0; \tag{2.24}$$

$$2) \sum_{p>2} p^{p+1} I_p^{1|p-1}(\Delta) < +\infty. \tag{2.25}$$

*then interaction, corresponding to this family of potentials, is also stable (superstable, strong superstable).*

# 3 Proof of Theorem 2.1

*Proof.* Let conditions of the theorem (2.1) hold and $\gamma \in \Gamma_0$. We can write $U^{(p)}(\gamma)$ in the following form:

$$\begin{aligned} U^{(p)}(\gamma) = &\sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq p}} \sum_{\{x_1,\ldots,x_p\} \subset \gamma_\Delta} V_p(x_1,\ldots,x_p) + \\ &+ \sum_{j=2}^{p} \sum_{\substack{k_l \geq 1, 1 \leq l \leq j, \\ k_1 + \cdots + k_j = p, \\ k_1 \leq \ldots \leq k_j}} \sum_{\substack{\{\Delta_1,\ldots,\Delta_j\} \subset \overline{\Delta_\lambda}, \\ |\gamma_{\Delta_r}| \geq 1, 1 \leq r \leq j}} \sideset{}{'}\sum_{\substack{\pi: k_{\pi(n)} \leq |\gamma_{\Delta_n}|, \\ 1 \leq n \leq j}} \times \\ &\times \sum_{\{x_1^{(1)},\ldots,x_{k_{\pi(1)}}^{(1)}\} \subset \gamma_{\Delta_1},\ldots,\{x_1^{(j)},\ldots,x_{k_{\pi(j)}}^{(j)}\} \subset \gamma_{\Delta_j}} V_p(x_1^{(1)},\ldots,x_{k_{\pi(j)}}^{(j)}). \end{aligned} \tag{3.1}$$

The first part of (3.1) includes the interaction of particles within every arbitrary cube $\Delta$, the second one does the same with particles, which are situated in different cubes

of $\overline{\Delta_\lambda}$ with $|\gamma_\Delta| \geq 1$. The 4-th group of sums in the second term of (3.1) is the sum over all different permutations (see def. 3) $\pi : (k_1, \ldots, k_j) \to (k_{\pi(1)}, \ldots, k_{\pi(j)})$ and all values $k_1, \ldots, k_j (k_1 \leq \ldots \leq k_j)$, $k_l \geq 1, l = \overline{1,j}$, $k_1 + \cdots + k_j = p$ with the restrictions $1 \leq k_{\pi(n)} \leq |\gamma_{\Delta_n}|, n = \overline{1,j}$. Let us explain this notation by simple example. Let the number of cubes, where there are particles for 7-potential be $j = 4$. The set of $k_i$ is (1, 2, 2, 2). We consider a set of cubes $\{\Delta_1, \ldots, \Delta_4\}$ such that $|\gamma_{\Delta_1}| = 1$, $|\gamma_{\Delta_2}| = 3$, $|\gamma_{\Delta_3}| = 2$, $|\gamma_{\Delta_4}| = 6$. As a result, all permutations $\pi$ such that $\pi(1) = 2$, $\pi(1) = 3$, $\pi(1) = 4$ are not allowed, i.e. $k_2 = k_3 = k_4 = 2 > |\gamma_{\Delta_1}|$. Using definitions (2.23) and (2.9) we can estimate (3.1) in the following way:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq p}} V_p^p(\Delta) C^p_{|\gamma_\Delta|} - \sum_{j=2}^{p} \sum_{\substack{k_l \geq 1,\, 1 \leq l \leq j, \\ k_1 + \cdots + k_j = p, \\ k_1 \leq \ldots \leq k_j}} \sum_{\substack{\{\Delta_1, \ldots, \Delta_j\} \subset \overline{\Delta_\lambda}, \\ |\gamma_{\Delta_r}| \geq 1,\, 1 \leq r \leq j}} \sum_{\substack{\pi:\, k_{\pi(n)} \leq |\gamma_{\Delta_n}|, \\ 1 \leq n \leq j}}{}^{\prime} \times \tag{3.2}$$
$$\times \left( \prod_{m=1}^{j} C^{k_{\pi(m)}}_{|\gamma_{\Delta_m}|} \right) \cdot \; I_p^{k_{\pi(1)}, \ldots, k_{\pi(j)}}(\Delta_1, \ldots, \Delta_j),$$

where $C_n^k = \frac{n!}{(n-k)!k!}$. Using inequalities: $\forall n \geq k \geq 1$, $\frac{n^k}{k^k} \leq C_n^k \leq \frac{n^k}{k!}$, we obtain:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq p}} \frac{V_p^p(\Delta)}{p^p} |\gamma_\Delta|^p - \sum_{j=2}^{p} \sum_{\substack{k_l \geq 1,\, 1 \leq l \leq j, \\ k_1 + \cdots + k_j = p, \\ k_1 \leq \ldots \leq k_j}} \sum_{\substack{\{\Delta_1, \ldots, \Delta_j\} \subset \overline{\Delta_\lambda}, \\ |\gamma_{\Delta_r}| \geq 1,\, 1 \leq r \leq j}} \times \tag{3.3}$$
$$\times \sum_{\substack{\pi:\, k_{\pi(n)} \leq |\gamma_{\Delta_n}|, \\ 1 \leq n \leq j}}{}^{\prime} I_p^{k_{\pi(1)}, \ldots, k_{\pi(j)}}(\Delta_1, \ldots, \Delta_j) \prod_{m=1}^{j} \frac{|\gamma_{\Delta_m}|^{k_{\pi(m)}}}{k_{\pi(m)}!}.$$

Let us consider the following inequality (see Appendix for the proof ):

$$\prod_{i=1}^{j} a_i^{m_i} \leq \frac{1}{m_1 + \cdots + m_j} \sum_{i=1}^{j} m_i \, a_i^{m_1 + \cdots + m_j} \leq \sum_{i=1}^{j} a_i^{m_1 + \cdots + m_j}, \tag{3.4}$$

where $a_1, \ldots, a_j \in \mathbb{R}_+; m_1, \ldots, m_j \in \mathbb{N}$. Using (3.4), we obtain:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq p}} \frac{V_p^p(\Delta)}{p^p} |\gamma_\Delta|^p - \sum_{j=2}^{p} \sum_{\substack{k_l \geq 1,\, 1 \leq l \leq j, \\ k_1 + \cdots + k_j = p, \\ k_1 \leq \ldots \leq k_j}} \sum_{\substack{\{\Delta_1, \ldots, \Delta_j\} \subset \overline{\Delta_\lambda}, \\ |\gamma_{\Delta_r}| \geq 1,\, 1 \leq r \leq j}} \times \tag{3.5}$$
$$\times \prod_{m=1}^{j} \frac{1}{k_m!} \sum_{\substack{\pi:\, k_{\pi(n)} \leq |\gamma_{\Delta_n}|, \\ 1 \leq n \leq j}}{}^{\prime} I_p^{k_{\pi(1)}, \ldots, k_{\pi(j)}}(\Delta_1, \ldots, \Delta_j) \sum_{i=1}^{j} |\gamma_{\Delta_i}|^p.$$

Taking into account, that the sum w.r.t. $\pi$ defined in (3.1) contains less number of terms, the the same one in (2.11), as it does not have the restrictions $k_{\pi(n)} \leq |\gamma_{\Delta_n}|$, $n = \overline{1,j}$ and using lemma 2, that is inequality (2.18), we obtain:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} \frac{V_p^p(\Delta)}{p^p}|\gamma_\Delta|^p - \sum_{j=2}^{p} \frac{j}{B(p;j)} \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}} |\gamma_\Delta|^p \sum_{\substack{k_l\geq 1,\, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\ldots\leq k_j}} I_p^{k_1|\{k_2,\ldots,k_j\}}(\Delta), \quad (3.6)$$

where

$$B(p;j) = \inf_{\substack{k_{\pi(t)}\geq 1,\, 1\leq t\leq p,\\ k_{\pi(1)}+\cdots+k_{\pi(j)}=p}} (k_{\pi(1)}! \cdot \ldots \cdot k_{\pi(j)}!)$$

Since $\max \frac{j}{B(p;j)} = p$, $2 \leq j \leq p$ and taking into account definitions 2, 3 and lemma 1, we deduce, that:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} \frac{V_p^p(\Delta)}{p^p}|\gamma_\Delta|^p - p\, I_p^{1|p-1} \sum_{\substack{\Delta\subset\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}} |\gamma_\Delta|^p. \quad (3.7)$$

Quantity of cubes with $|\gamma_\Delta| = k$ is not more than $\frac{|\gamma|}{k}$. Due to this, the following estimate holds:

$$\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}} |\gamma_\Delta|^p = \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} |\gamma_\Delta|^p + \sum_{k=1}^{p-1} \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|=k}} |\gamma_\Delta|^p \leq \quad (3.8)$$

$$\leq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} |\gamma_\Delta|^p + \sum_{k=1}^{p-1} k^{p-1}|\gamma|.$$

Using (3.7) and (3.8), we obtain the final estimate of $U^{(p)}(\gamma)$:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} |\gamma_\Delta|^p \left(\frac{V_p^p(\Delta)}{p^p} - pI_p^{1|p-1}\right) - pI_p^{1|p-1} \sum_{k=1}^{p-1} k^{p-1}|\gamma|. \quad (3.9)$$

Let us take into account the following obvious estimate:

$$\sum_{p>2} p\, I_p^{1|p-1} \sum_{k=1}^{p-1} k^{p-1} < \sum_{p>2} p^{p+1}\, I_p^{1|p-1}. \quad (3.10)$$

The condition of stability (superstability, strong superstability) (2.24) - (2.25) follows directly from the last estimates (3.9), (3.10) with

$$B = B_2 + \sum_{p>2} p^{p+1}\, I_p^{1|p-1}, \quad (3.11)$$

where $B_2$ is taken from the condition of superstability(strong superstability)of 2-body part of interaction.

## 4 Example of many-body interaction

First consider one-dimensional case ($d = 1$).

**Example 1.** *Let $V$ be a many-body interaction, specified by a family of* p*-body potentials* $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$*:*

$$V_p(x_1, \ldots, x_p) = \frac{A_p}{\left(\sum\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{m(p)}} - \frac{B_p}{\left(\sum\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{n(p)}}, \tag{4.1}$$

$$A_p > 0, B_p > 0;\ m(p) > n(p),\ n(p) > p - 1.$$

Prove that such a family of potentials satisfies assumptions **A1-A4** and write down the conditions on $A_p, B_p$, that ensure superstability of interaction.

Verification of assumptions **A1-A3** is obvious. Let us analyze the last assumption **A4** for the family of p-body potentials (4.1). Denote the following area, which will be used in our future estimates:

$$Q_{I_p}(x_1) = \left\{ \{x_2, \ldots, x_p\} \in \mathbb{R}^{d(p-1)} \mid \sum_{1\leq i<j\leq p} |x_i - x_j| \geq \left(\frac{A_p}{B_p}\right)^{\frac{1}{m(p)-n(p)}} \right\} \tag{4.2}$$

Using (4.2) we can write

$$I_p = \int_{\left(\mathbb{R}^d\right)^{p-1}} \left|V_p^-\ (x_1, \ldots, x_p)\right|\ dx_2 \cdot \ldots \cdot dx_p = \tag{4.3}$$

$$= \int_{Q_{I_p}(x_1)} \left( \frac{B_p}{\left(\sum\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{n(p)}} - \frac{A_p}{\left(\sum\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{m(p)}} \right) dx_2 \cdot \ldots \cdot dx_p.$$

Consider the following estimates of sum $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ in (4.1). The minimum of $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ is reached, if $p-2$ particles coincide. The maximum of $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ is reached, if $\left[\frac{p}{2}\right]$ particles are situated at one point and the rest of them are situated at another one:

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \geq (p-1) \max_{1\leq i<j\leq p} |x_i - x_j|; \tag{4.4}$$

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \leq \left(p - \left[\frac{p}{2}\right]\right) \left[\frac{p}{2}\right] \max_{1\leq i<j\leq p} |x_i - x_j|. \tag{4.5}$$

**Remark 4.** *In d-dimensional case the estimate* (4.4) *is also true, but the second one* (4.5) *requires the following modification:*

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \leq \frac{p(p-1)}{2} \max_{1\leq i<j\leq p} |x_i - x_j|.$$

Taking into account (4.4), (4.5) we can rewrite (4.3) in the following form:

$$I_p \le \int_{Q'_{I_p}(x_1)} \left( \frac{B'_p}{\left(\max\limits_{1\le i<j\le p} |x_i - x_j|\right)^{n(p)}} - \frac{A'_p}{\left(\max\limits_{1\le i<j\le p} |x_i - x_j|\right)^{m(p)}} \right) dx_2 \cdot \ldots \cdot dx_p, \quad (4.6)$$

where $A'_p = \frac{A_p}{((p-[\frac{p}{2}])[\frac{p}{2}])^{m(p)}}, \quad B'_p = \frac{B_p}{(p-1)^{n(p)}}$,

$$Q'_{I_p}(x_1) = \left\{ \{x_2, \ldots, x_p\} \in \mathbb{R}^{d(p-1)} \mid \max_{1\le i<j\le p} |x_i - x_j| \ge \left(\frac{A'_p}{B'_p}\right)^{\frac{1}{m(p)-n(p)}} \right\} \quad (4.7)$$

For definiteness we will assume, that: $x_1 = 0$. There are two types of configurations:

1) $\mathrm{diam}(\{x_1, \ldots, x_p\}) = \mathrm{dist}(x_i; x_j)$, for some $x_i, x_j$ and $x_i < x_1 < x_j$;
2) $\mathrm{diam}(\{x_1, \ldots, x_p\}) = \mathrm{dist}(x_1; x_j)$, for some $x_j$

In accordance with these 2 cases we can rewrite (4.6) in the following form:

$$\begin{aligned} I_p \le\; & A^2_{p-1} \int_{-\infty}^{-\left(\frac{A'_p}{B'_p}\right)^{\frac{1}{m(p)-n(p)}}} dx_2 \cdot \int_0^{+\infty} \left( \frac{B'_p}{(x_p - x_2)^{n(p)}} - \frac{A'_p}{(x_p - x_2)^{m(p)}} \right) dx_p \times \\ & \times \int_{x_2}^{x_p} dx_3 \cdots \int_{x_2}^{x_p} dx_{p-1} + \\ & + A^2_{p-1} \int_{-\left(\frac{A'_p}{B'_p}\right)^{\frac{1}{m(p)-n(p)}}}^{0} dx_2 \cdot \int_{x_2 + \left(\frac{A'_p}{B'_p}\right)^{\frac{1}{m(p)-n(p)}}}^{+\infty} \left( \frac{B'_p}{(x_p - x_2)^{n(p)}} - \frac{A'_p}{(x_p - x_2)^{m(p)}} \right) dx_p \times \\ & \times \int_{x_2}^{x_p} dx_3 \cdots \int_{x_2}^{x_p} dx_{p-1} + \\ & + 2(p-1) \int_{\left(\frac{A'_p}{B'_p}\right)^{\frac{1}{m(p)-n(p)}}}^{+\infty} \left( \frac{B'_p}{x_p^{n(p)}} - \frac{A'_p}{x_p^{m(p)}} \right) dx_p \int_0^{x_p} dx_2 \cdots \int_0^{x_p} dx_{p-1}. \end{aligned} \quad (4.8)$$

The first and the second integrals in (4.8) refer to the case 1). In (4.8) $A^2_{p-1}$ is a number of all possible pairs $(x_i, x_j)$, $1 < i < j \le p$ with respect to their order: $A^k_n = \frac{n!}{(n-k)!}$. The third integral in (4.8) refers to the case 2). In (4.8) $2(p-1)$ is a number of $x_i$, $1 < i \le p$ with respect to its right or left position from the origin. Under the condition

$m(p) > n(p),\ n(p) > p-1$ all integrals (4.8) converge and finally:

$$I_p \leq \left(\frac{A_p^{'}}{B_p^{'}}\right)^{\frac{p-1}{m(p)-n(p)}} \left(A_{p-1}^2\left(\frac{A_p^{'}}{(p-2-m(p))\left(\frac{A_p^{'}}{B_p^{'}}\right)^{\frac{m(p)}{m(p)-n(p)}}} - \frac{B_p^{'}}{(p-2-n(p))\left(\frac{A_p^{'}}{B_p^{'}}\right)^{\frac{n(p)}{m(p)-n(p)}}} + \right.\right.$$

$$\left. + \frac{A_p^{'}(-1)^{p-2-m(p)}}{(p-2-m(p))(p-1-m(p))} - \frac{B_p^{'}(-1)^{p-2-n(p)}}{(p-2-n(p))(p-1-n(p))}\right) +$$

$$\left.+ 2(p-1)\left(\frac{A_p^{'}}{(p-1-m(p))\left(\frac{A_p^{'}}{B_p^{'}}\right)^{\frac{m(p)}{m(p)-n(p)}}} - \frac{B_p^{'}}{(p-1-n(p))\left(\frac{A_p^{'}}{B_p^{'}}\right)^{\frac{n(p)}{m(p)-n(p)}}}\right)\right). \tag{4.9}$$

Consequently, the assumption **A4** holds. We obtain from (2.25), (4.9) the condition on $A_p, B_p$, that ensures supersable interaction:

$$\sum_{p\geq 2} p^{p+1}\, I_p < +\infty, \tag{4.10}$$

so we can put for example:

$$n(p) = p,\ m(p) = p+1,\ A_p = \left(\left(p - \left[\frac{p}{2}\right]\right)\left[\frac{p}{2}\right]\right)^{p+1},\ B_p < \frac{p^{\frac{p-4}{2}-\varepsilon}}{2^{\frac{2p-1}{2}}},\ \varepsilon > 0. \tag{4.11}$$

Let us find an estimate of $U^{(3)}(\gamma)$. For simplicity: $A_3 = B_3 = 1$; $m(3) = 12$, $n(3) = 6$. Then: $A_3^{'} = 1/2^{12}$, $B_3^{'} = 1/2^6$. Using (4.9), we obtain: $I_3 \leq \frac{477}{20480}$. Taking into account Remark 2, we conclude, that: $I_3^{1|2}(\Delta)\lambda^2 \to \frac{477}{20480}$, $\lambda \to 0$. Now estimate $V_3^3(\Delta)$, where $\Delta \in \overline{\Delta_\lambda}$, $V_3^3(\Delta) \geq 0$, for any $\{x_1, \cdots, x_p\} \subset \Delta$. It follows from (4.4), (4.5), that:

$$V_p^p(\Delta) \geq \frac{A_p^{'}}{\left(\max\limits_{1\leq i<j\leq p}|x_i - x_j|\right)^{m(p)}} - \frac{B_p^{'}}{\left(\max\limits_{1\leq i<j\leq p}|x_i - x_j|\right)^{n(p)}}. \tag{4.12}$$

Function in the right part of (4.12) achieves its minimum in the cubic area, if: $\max\limits_{1\leq i<j\leq p}|x_i - x_j| = \lambda$, where $\lambda$ is a rib of $\Delta$. Using (4.12), we obtain: $V_3^3(\Delta) \geq \frac{1}{4096\lambda^{12}} - \frac{1}{64\lambda^6}$. We have finally from (3.9):

$$U^{(3)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 3}} |\gamma_\Delta|^3 \left(\frac{1}{110592\lambda^{12}} - \frac{1}{1728\lambda^6} - \frac{1431}{20480\lambda^2}\right) - \frac{1431}{4096\lambda^2}|\gamma|.$$

If $\lambda \leq 0.29874$, then condition (2.24) of Theorem 2.1 holds.

Now, consider the general case $d > 1$ with $B_p^{\prime} > 0$ and $n(p) > (p-1)d$. It is clear that

$$\left|V_p^{-}\ (x_1, \ldots, x_p)\right| \leq \frac{B_p^{\prime}}{\left(\max\limits_{1\leq i<j\leq p}|x_i - x_j|\right)^{n(p)}},$$

Prove that it satisfies assumption **A4**.

*Proof.* Consider such a ball $B(0;R_0)$ with center in the origin and a radius $R_0$ that $V_p\left(x_1,\ldots,x_p\right)\geq 0$ for any $p\geq 2$ and put $x_1=0$. As in the case $d=1$ consider 2 cases:

$$\begin{aligned}&1)\,\mathrm{diam}(\{x_1,\ldots,x_p\})=\mathrm{dist}(x_i;x_j),\\&0\in B\left(\frac{x_i+x_j}{2};\,\frac{|x_j-x_i|}{2}\right),1<i\leq p,1<j\leq p;\\&2)\;\mathrm{diam}(\{x_1,\ldots,x_p\})=\mathrm{dist}(0;x_j),\;1<j\leq p.\end{aligned}$$

In accordance with these 2 cases one can write the following estimate:

$$\int_{\left(\mathbb{R}^d\right)^{p-1}}\left|V_p^-\;(0,x_2,\ldots,x_j)\right|\;dx_2\cdots dx_p\leq I_p^{(1)}+I_p^{(2)};$$

$$I_p^{(1)}\leq B_p'\cdot C_{p-1}^2\int\limits_{\substack{\frac{|x_2+x_p|}{2}\leq\frac{|x_p-x_2|}{2},\\|x_p-x_2|>2R_0}}\frac{dx_2\cdot dx_p}{|x_p-x_2|^{n(p)}}\times$$

$$\times\int\limits_{\left|\frac{x_p+x_2}{2}-x_3\right|\leq\frac{|x_p-x_2|}{2}}dx_3\cdots\int\limits_{\left|\frac{x_p+x_2}{2}-x_{p-1}\right|\leq\frac{|x_p-x_2|}{2}}dx_{p-1},\tag{4.13}$$

$$I_p^{(2)}\leq B_p'(p-1)\int\limits_{|x_p|>2R_0}\frac{dx_p}{|x_p|^{n(p)}}\cdot\int\limits_{\left|\frac{x_p}{2}-x_2\right|\leq\frac{|x_p|}{2}}dx_2\cdots\int\limits_{\left|\frac{x_p}{2}-x_{p-1}\right|\leq\frac{|x_p|}{2}}dx_{p-1}.\tag{4.14}$$

The first integral $I_p^{(1)}$ and the second $I_p^{(2)}$ refer to cases 1) and 2) respectively. In (4.13) $C_{p-1}^2$ is a quantity of all possible pairs $\{x_i,x_j\},\;1<i<j\leq p$ without respect to their order. In (4.14) $p-1$ is a quantity of $x_j,\;1<j\leq p$. The case of d-dimensional space differs from 1-dimensional in the way, that the order of the "remotest" variables $x_i,x_j$ is neglected. Let's take into account that a volume of d-dimensional ball $B(a,R)$ is:

$$\int_{|x-a|\leq R}dx=\frac{2\pi^{\frac{d}{2}}R^d}{d\,\Gamma\left(\frac{d}{2}\right)}.\tag{4.15}$$

Using (4.15) one can rewrite (4.13), (4.14) in the following form:

$$I_p^{(1)}\leq B_p'\,C_{p-1}^2\left(\frac{\pi^{\frac{d}{2}}}{2^{d-1}d\,\Gamma\left(\frac{d}{2}\right)}\right)^{p-3}\int\limits_{\substack{\frac{|x_2+x_p|}{2}\leq\frac{|x_p-x_2|}{2},\\|x_p-x_2|>2R_0}}\frac{dx_2\cdot dx_p}{|x_p-x_2|^{n(p)-(p-3)d}},\tag{4.16}$$

$$I_p^{(2)}\leq B_p'\,(p-1)\left(\frac{\pi^{\frac{d}{2}}}{2^{d-1}d\,\Gamma\left(\frac{d}{2}\right)}\right)^{p-2}\int\limits_{|x_p|>2R_0}\frac{dx_p}{|x_p|^{n(p)-(p-2)d}}.\tag{4.17}$$

In (4.16) we do the following substitution of variables: $\{x_2; x_p\} \to \{x_2; t\}$, $t = x_p - x_2$. We obtain:

$$I_p^{(1)} \le B'_p\, C_{p-1}^2 \left( \frac{\pi^{\frac{d}{2}}}{2^{d-1} d\, \Gamma\left(\frac{d}{2}\right)} \right)^{p-3} \int\limits_{|t|>2R_0} \frac{dt}{|t|^{n(p)-(p-3)d}} \int\limits_{|x_2+\frac{t}{2}|\le\frac{|t|}{2}} dx_2 =$$

$$= B'_p\, C_{p-1}^2 \left( \frac{\pi^{\frac{d}{2}}}{2^{d-1} d\, \Gamma\left(\frac{d}{2}\right)} \right)^{p-2} \int\limits_{|t|>2R_0} \frac{dt}{|t|^{n(p)-(p-2)d}} \tag{4.18}$$

Using generalized spherical coordinates, we deduce from (4.17) and (4.18) that:

$$I_p^{(1)} \le B'_p\, C_{p-1}^2 2^d d \left( \frac{\pi^{\frac{d}{2}}}{2^{d-1} d\, \Gamma\left(\frac{d}{2}\right)} \right)^{p-1} \int_{2R_0}^{+\infty} \frac{dr}{r^{n(p)+(1-p)d+1}}, \tag{4.19}$$

$$I_p^{(2)} \le B'_p\, (p-1) 2^d d \left( \frac{\pi^{\frac{d}{2}}}{2^{d-1} d\, \Gamma\left(\frac{d}{2}\right)} \right)^{p-1} \int_{2R_0}^{+\infty} \frac{dr}{r^{n(p)+(1-p)d+1}}. \tag{4.20}$$

If $n(p) + (1-p)d > 0$ integrals (4.19) and (4.20) converge and finally:

$$\int_{(\mathbb{R}^d)^{p-1}} \left| V_p^-\,(0, x_2, \ldots, x_p) \right|\, dx_2 \cdots dx_p \le \frac{B'_p\, d\, \left(p - 1 + C_{p-1}^2\right)}{2^{n(p)-pd}(n(p) + (1-p)d) R_0^{n(p)+(1-p)d}} \times$$
$$\times \left( \frac{\pi^{\frac{d}{2}}}{2^{d-1} d\, \Gamma\left(\frac{d}{2}\right)} \right)^{p-1}. \tag{4.21}$$

In this case condition on $B'_p$, which ensures superstability, can be easily obtained in a similar way as in the example (4.1).

■

# 5 Appendix

For arbitrary $a_1, \ldots, a_j \in \mathbb{R}_+; m_1, \ldots, m_j \in \mathbb{N}$ the following inequality holds:

$$\prod_{i=1}^{j} a_i^{m_i} \le \frac{1}{m_1 + \cdots + m_j} \sum_{i=1}^{j} m_i\, a_i^{m_1+\cdots+m_j}. \tag{5.1}$$

*Proof.* Let's prove (5.1) in the case of $j = 2$.
Let $j = 2, m_2 = 1$ in (5.1). Then the following is true:

$$a_1^{m_1} a_2 \le \frac{m_1}{m_1+1} a_1^{m_1+1} + \frac{1}{m_1+1} a_2^{m_1+1}. \tag{5.2}$$

We will prove this fact by induction. If $m_1 = 1$, then inequality (5.2) transforms into trivial: $a_1 a_2 \le \frac{a_1^2}{2} + \frac{a_2^2}{2}$. We suppose, that (5.2) is true for $m_1 - 1$. Taking into account this statement, the following range of estimates holds:

$$a_1^{m_1} a_2 = a_1^{m_1-1}(a_1 a_2) \le \frac{a_1^{m_1+1}}{2} + \frac{a_1^{m_1-1} a_2^2}{2} \le \frac{a_1^{m_1+1}}{2} + \frac{a_2}{2}\left[\frac{m_1-1}{m_1} a_1^{m_1} + \frac{1}{m_1} a_2^{m_1}\right]. \quad (5.3)$$

From (5.3) we obtain:

$$\frac{m_1+1}{2m_1} a_1^{m_1} a_2 \le \frac{1}{2} a_1^{m_1+1} + \frac{1}{2m_1} a_2^{m_1+1}. \quad (5.4)$$

The estimate (5.2) follows directly from (5.4). We deduce from (5.2):

$$\begin{aligned} a_1^{m_1} a_2^{m_2} &= a_2^{m_2-1}(a_1^{m_1} a_2) \le \frac{m_1}{m_1+1} a_1^{m_1+1} a_2^{m_2-1} + \frac{1}{m_1+1} a_2^{m_1+m_2} \le \quad (5.5) \\ &\le a_2^{m_2-2} \frac{m_1}{m_1+1}\left[\frac{m_1+1}{m_1+2} a_1^{m_1+2} + \frac{1}{m_1+2} a_2^{m_1+2}\right] + \frac{1}{m_1+1} a_2^{m_1+m_2} = \\ &= \frac{m_1}{m_1+2} a_1^{m_1+2} a_2^{m_2-2} + \frac{2}{m_1+2} a_2^{m_1+m_2} \le \ldots \le \frac{m_1}{m_1+m_2} a_1^{m_1+m_2} + \frac{m_2}{m_1+m_2} a_2^{m_1+m_2}. \end{aligned}$$

The estimate (5.1) is proven for $j = 2$. Let (5.1) be true for $j - 1$. Using this, we obtain the following range of estimates:

$$\begin{aligned} \prod_{i=1}^{j} a_i^{m_i} &\le \frac{1}{m_1 + \cdots + m_{j-1}} \sum_{i=1}^{j-1} m_i\, a_i^{m_1+\cdots+m_{j-1}}\ a_j^{m_j} \le \quad (5.6) \\ &\le \frac{1}{m_1 + \cdots + m_{j-1}} \sum_{i=1}^{j-1} \left[\frac{m_i(m_1 + \cdots + m_{j-1})}{m_1 + \cdots + m_j} a_i^{m_1+\cdots+m_j} + \frac{m_i m_j}{m_1 + \cdots + m_j} a_j^{m_1+\cdots+m_j}\right]. \end{aligned}$$

One can deduce from the last estimate in (5.6):

$$\begin{aligned} \prod_{i=1}^{j} a_i^{m_i} &\le \frac{1}{m_1 + \cdots + m_j} \sum_{i=1}^{j-1} m_i\, a_i^{m_1+\cdots+m_j} + \quad (5.7) \\ &+ \frac{m_j}{(m_1 + \cdots + m_{j-1})(m_1 + \cdots + m_j)}\, a_j^{m_1+\cdots+m_j} \sum_{i=1}^{j-1} m_i. \end{aligned}$$

The estimate (5.1) is a consequence of (5.7). The end of the proof.

■

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